UNITED STATES AIR FORCE

AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Air Force Institute of Technology Wright-Patterson Air Force Base, Ohio	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE
A MONTE CARLO TECHNIQUE USING COMPONENT FAILURE TEST DATA TO APPROXIMATE RELIABILITY CONFIDENCE LIMITS OF SYSTEMS WITH COMPONENTS CHARACTERIZED BY THE WEIBULL DISTRIBUTION

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
AFIT Thesis

5. AUTHOR(S) (First name, middle initial, last name)
Robert G. Lannon
Capt USAF

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March 1972	53	15

8a. CONTRACT OR GRANT NO. b. PROJECT NO. N/A c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) GAM/MA/72-2 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

Approved for public release; IAW AFR 190-17 Keith A. Williams Keith A. Williams, 1st Lt., USAF Acting Director of Information	12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

This paper develops a Monte Carlo technique which, with a digital computer, determines confidence limits for system reliability of complex systems containing components characterized by the Weibull distribution. The component distribution shape and scale parameters are estimated by the method of maximum likelihood from component failure times while the location parameter is assumed known. The asymptotic distribution of these maximum likelihood estimators and a Monte Carlo simulation are used to determine confidence limits on system reliability. As an example, confidence limits are calculated for two systems of up to eight components in combinations of series and parallel configurations using 99, 499, 999, and 2999 simulations. Accuracy of the confidence limits is found to be satisfactory after being checked by a method using a double Monte Carlo technique.

DD FORM 1 NOV 65 1473

Unclassified
Security Classification

Unclassified
Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Monte Carlo Method						
Maximum Likelihood Estimators						
Reliability Estimates						
Weibull Density Functions						
Confidence Limits of Reliability						
Forecasting Reliability						
Computerized Simulation						

Unclassified
Security Classification

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THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

by

Robert G. Lannon, B.M.S.
Captain USAF

March 1972

Preface

This thesis is a continuation of previous work done at the Air Force Institute of Technology on Monte Carlo techniques of reliability prediction. It is hoped that the method proposed here may find use in reliability analysis of complex systems and/or in situations where limited failure test data is available to the reliability engineer.

I wish to thank my thesis advisor, Professor Albert H. Moore, for suggesting the topic and giving guidance and encouragement throughout this project.

I also wish to express my gratitude to Dr. H. Leon Harter who gave valuable advise during this project. The FORTRAN programs in Appendix A and Appendix C are revisions of programs which he originally wrote and kindly supplied to me.

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Abstract

This paper develops a Monte Carlo technique which, with a digital computer, determines confidence limits for system reliability of complex systems containing components characterized by the Weibull distribution. The component distribution shape and scale parameters are estimated by the method of maximum likelihood from component failure times while the location parameter is assumed known. The asymptotic distribution of these maximum likelihood estimators and a Monte Carlo simulation are used to determine confidence limits on system reliability. As an example, confidence limits are calculated for two systems of up to eight components in combinations of series and parallel configurations using 99, 499, 999, and 2999 simulations. Accuracy of the confidence limits is found to be satisfactory after being checked by a method using a double Monte Carlo technique which assumes values for parameters of the component distributions and generates component failure times to be used in the proposed Monte Carlo technique. Central processing time for this technique on the CDC 6600 digital computer is found to be less than two minutes for the examples given. There is no limit to the number of components or the configurations of the components for the systems to which the method can be applied. The FORTRAN IV computer programs used are given along with flowcharts to facilitate reading the programs.

A MONTE CARLO TECHNIQUE USING COMPONENT FAILURE TEST DATA TO APPROXIMATE RELIABILITY CONFIDENCE LIMITS OF SYSTEMS WITH COMPONENTS CHARACTERIZED BY THE WEIBULL DISTRIBUTION

I. Introduction

The Problem

The purpose of this thesis is to determine whether accurate interval estimation of reliability for complicated systems, containing dissimilar components characterized by the Weibull distribution, can be obtained by using maximum likelihood estimates and a Monte Carlo simulation.

Significance. Systems containing many components become increasingly complex and costly. Complicated systems costing thousands of dollars can become inoperative due to the failure of a single connection. Because of this, it is important to determine system reliability as a function of the component reliabilities which make up the system. This can be done fairly easily for some few types of combinations of components which are from certain parent populations such as an electrical circuit made up of components characterized by the exponential distribution. When the components are dissimilar, no simple analytical expression can be used to compute system reliability.

In addition, predictions of reliability should be based on test data. However, as system reliability requirements

increase, testing a system to destruction can become expensive, both in time needed and money spent.

A method that circumvents this problem uses limited test data to calculate maximum likelihood estimates of the parameters of the failure distribution. These estimates are used, along with a random number generator, in a Monte Carlo technique which uses the asymptotic distribution of the maximum likelihood estimators to generate more point samples of system reliability. These point samples are used to determine an interval estimate of system reliability with the associated confidence level.

Definitions. Reliability is the probability that a system is still operating at time t. If T is the time to failure or life length of a system or component, the reliability at time t or R(t) is given by $R(t) = P(T > t)$, where P means "probability of".

The Weibull density function is defined as

$$f(t;C,\theta,K) = \left[\frac{K(t-C)^{K-1}}{\theta^K}\right] \exp\left\{-\left[\frac{t-C}{\theta}\right]^K\right\} \qquad (1)$$

$$\theta,\ K > 0,\quad C \leq t$$

$$= 0 \text{ elsewhere}$$

where θ = the scale parameter, K = the shape parameter, and C = location parameter. The scale parameter affects the dispersion of the random variable t about its mean. The shape parameter determines whether the hazard function is

increasing, decreasing, or time invariant; while the location parameter determines the point of origin.

Reliability, for the Weibull density function, is

$$R(t) = \exp\left\{-\left[\frac{t-C}{\theta}\right]^{K}\right\} \qquad (2)$$

Pseudo-random numbers are generated from a series of numbers which repeats itself with long periods between repetitions. These numbers are suitably random if the repetition periods are greater than the number of digits desired and if there is a small enough correlation between digits (Ref 15:257).

The maximum likelihood estimator (MLE) is the estimator which maximizes the likelihood function with respect to the parameter being estimated. The likelihood function is the joint density of a sample of n random variables and is expressed by

$$L(x_1,x_2..x_n;\theta) = \prod_{i=1}^{n} f(x_i;\theta) \qquad (3)$$

where $L(x_i:\theta)$ = the likelihood function.

If it is desired to determine $\hat{\theta}$ (the MLE for the parameter θ), normally it is only necessary to take the partial derivative of the likelihood function with respect to θ, set it equal to zero and solve for θ which then becomes $\hat{\theta}$. By using the MLE for each parameter and the expression for reliability for a given failure density, it is possible to arrive at a point estimate for reliability.

The method in this paper generates intervals from a random sample of system reliabilities and this is called a random interval. If reliability falls within any given interval, there is associated with the interval a confidence level. For example, if it is said the reliability of a system lies between 0.6 and 0.8 at the 95% confidence level, it is meant $P(0.6 \leq R(t) \leq 0.8) = .95$ or the random interval 0.6 - 0.8 would contain the true system reliability 95% of the time if the system were tested many times. The interval is called a confidence interval, which has 0.6 as a lower confidence limit and 0.8 as an upper confidence limit.

A Monte Carlo simulation is one in which a random value or number is picked from all possible numbers that can be described by a specific density function.

Analysis

Assumptions. This thesis will concern itself with systems which have components described by the Weibull density function. It is assumed that the components have previously been determined to be Weibull or that the Weibull density function adequately models the components in the system. This paper does not provide a method to test for the Weibull density function though these tests do exist and are described in other references. One of the easiest ways to test for the Weibull distribution is to use Weibull graph paper (Ref 5:170-173). Once it is determined or suspected the components are Weibull, the methods in this paper can be

used to estimate the parameters of the Weibull distribution and then to generate interval estimates of system reliability. The Weibull distribution is rather versatile in that, depending on the value of the shape parameters, the Weibull distribution can be used to model components with decreasing, increasing or constant failure rates or hazard functions. A special case of the Weibull distribution is the exponential distribution which occurs when the shape parameter, K, is equal to one.

It is assumed that the components of the system being analyzed fail independently. This allows use of standard formulas for components connected in series or in parallel. For instance, if a system is composed of two components connected in series, the reliability can be expressed by

$$R_s(t) = [R_1(t)][R_2(t)] \tag{4}$$

where $R_1(t)$ = reliability of component number one and, $R_2(t)$ = reliability of component number two.

Likewise, for a system made up of two components in parallel the reliability can be expressed by

$$\begin{aligned} R_s(t) &= 1 - [1 - R_1(t)][1 - R_2(t)] \\ &= 1 - Q_1 Q_2 \end{aligned} \tag{5}$$

where $Q_i = 1 - R_i(t)$.

More complex systems can be reduced to combinations of series and/or parallel configurations by use of Bayes'

theorem or the Boolean Disjunctive theorem. In other cases where the system reliability dependence on component reliability is known and can be expressed by other equations, the independent failure assumption is not necessary.

It is assumed that the location parameter is known or can be set equal to zero for all Weibull component distributions in this paper.

<u>Standards</u>. The only true criterion that is applied to test the acceptability of the method in this paper is analysis of the result. Does the computed confidence interval contain the actual value of system reliability the required number of times? To check the validity of the method, the values of all parameters were assumed for each of the Weibull distributions of the components. Pseudo-random component failure times were generated for each of the components and the method in this paper applied to generate many confidence intervals. Since the true system reliability was known, a check was made to see if the true reliability was in the calculated intervals the required number of times. This process is described in Chapter II.

For the estimation of the parameters from the generated component failure times, variation from the true value of the parameter in the second digit was deemed acceptable.

<u>Background</u>. Levy and Moore developed a method for determining system reliability confidence limits using a Monte Carlo method where the component failure distributions (exponential, normal, lognormal, Gamma, or Weibull) with

location and shape parameters were known (Ref 10). Lutton extended this method by using the asymptotic distribution of the maximum likelihood estimators of, among others, the Weibull distribution, location parameter known (Ref 11).

Harter and Moore developed two procedures relevant to this paper. They used an iterative procedure to determine maximum likelihood estimates of the parameters of the Weibull distribution from complete and censored samples (Ref 8). They also developed a method to determine the asymptotic variances and covariances of the maximum likelihood estimators of the parameters of the Weibull distribution from samples that were censored or complete (Ref 7). This paper presents a technique in which the sample of failure times for each component is used, along with the asymptotic distribution of the Weibull parameters, to generate a confidence interval with the associated confidence level.

Approach. The method and associated computer program in this paper:

1. Calculate the MLE of the component shape and scale parameters from the component failure times.
2. Determine the variance-covariance matrix of the asymptotic normal distribution of the shape and scale parameters.
3. Generate a sample of the component shape and scale parameters.
4. Use sample shape and scale parameters, along with known location parameter, to calculate a point

estimate of component reliability for each component.

5. Calculate a point sample of system reliability from the point samples of component reliabilities.

6. Repeat steps 3-5 to obtain many samples of system reliability.

7. Order the system reliability samples to obtain an interval sample of system reliability from which the confidence limits are determined at a given confidence level.

II. Procedure

Estimation of Component Parameters

The computer program for finding the MLE of the parameters of the Weibull distribution from complete or censored samples is given in Appendix A. The procedure, as developed by Harter and Moore, uses the natural logarithm of the likelihood function of the remaining order statistics which is

$$L_r = \ln n! - \ln(n-m)! - \ln r! + (m-r)(\ln K - K \ln \theta) +$$

$$(K-1) \sum_{i=r+1}^{m} \ln(x_i - C) - \sum_{i=r+1}^{m} \left[\frac{(x_i - C)}{\theta} \right]^K -$$

$$(n-m) \left[\frac{x_m - C}{\theta} \right]^K + r \ln \left\{ 1 - \exp\left[\frac{-(x_{r+1} - C)^K}{\theta^K} \right] \right\} \quad (6)$$

where $(m-r)$ is the number of order statistics $x_{r+1}, x_{r+2}, \ldots, x_m$ and n is the size of the sample before censoring.

The partial derivatives of L_r with respect to each of the three parameters are set equal to zero and the three resulting equations solved simultaneously. The iterative procedure for the simultaneous solution of the equations uses an initial estimate of the parameter (which is fed into the computer along with the failure times) and the rule of false position to determine a new value of the parameter from the appropriate likelihood equation into which the latest estimates (or a known value) of the other two parameters have been substituted (Ref 8:641).

The program in Appendix A has been converted into a subroutine and is called for analysis of each component. When this is done, the MLE of the shape and scale parameters are determined for all components in the system.

Generating Sample System Reliabilities

The asymptotic joint distribution of the estimators with the location parameter known is bivariate normal and is given by

$$f(Y) = \frac{1}{|V|^{1/2}(2\pi)} [\exp(- \tfrac{1}{2} Y'V^{-1}Y)] \tag{7}$$

where Y = two component vector of estimates of the parameters

Y' = transpose of Y

V = asymptotic variance-covariance matrix

V^{-1} = inverse of V

$|V|$ = determinant of V

Let $Z = V^{-1/2}Y$ so that $Y = V^{1/2}Z$ and, because V is symmetric, $Y' = Z'V^{1/2}$. Then

$$f(Z) = \frac{1}{2\pi} [\exp(- \tfrac{1}{2} Z'V^{1/2}V^{-1}V^{1/2}Z)] \tag{8}$$

$$= \frac{1}{2\pi} [\exp(- \tfrac{1}{2} Z'Z)] \tag{9}$$

$$= \frac{1}{2\pi} [\exp(Z_1^{\,2} + Z_2^{\,2})] \tag{10}$$

so that the density function of Z is made up of independent variates of the normal distribution with mean zero and variance 1 (Ref 14). Thus, to obtain a sample of the joint distribution of the estimators, the following formula is used:

$$Y' = V^{1/2}Z + Y \tag{11}$$

where Y' = 2 X 1 vector of the biased random samples of the estimates of the shape and scale parameters.

V = 2 X 2 variance-covariance matrix.

Z = 2 X 1 vector of pseudo-random numbers from the standard normal distribution.

Y = 2 X 1 vector of the maximum likelihood estimates of the two parameters (as found in the previous section).

or

$$\begin{bmatrix} \hat{\theta}_{sample} \\ \hat{K}_{sample} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} \hat{\theta}_{MLE} \\ \hat{K}_{MLE} \end{bmatrix} \tag{12}$$

In formula (12) the MLE of θ and K are used as the marginal means of the bivariate normal distribution of the parameter estimates.

The random samples of the component parameter estimates were substituted into the expression for reliability for each component to get random values of component reliabilities. The were then substituted into the expression for system

reliability, one value for each component, to generate random values of system reliability.

The program which generated the system reliabilities is given in Appendix B, along with a flow chart of the program. A detailed explanation of the evaluation of $V^{1/2}$ is also given in Appendix B.

The Variance-Covariance Matrix

The natural logarithm of the likelihood function for a sample of size n from a Weibull population where the lowest r and the highest n-m sample values have been censored is given in Ref 7. The elements of the information matrix are found by taking the limits, as $n \to \infty$, of the negatives of the expected values of the second partial derivatives of the likelihood function with respect to each of the parameters. Let k_{ij} be the elements of a matrix B, formed from the elements of the information matrix (s_{11}, $s_{12} = s_{21}$, and s_{22}), where E[...] is the conditional expectation given z_{r+1} and z_m. Then

$$k_{11} = \frac{v_{11}\theta^2}{n} = \lim pr\ E\left[-\left(\frac{\theta^2}{n}\right)\left(\frac{\partial^2 L}{\partial\theta^2}\right)\right]$$

$$= -Kp + K(K+1)[\Gamma(2;\hat{z}_m^K) - \Gamma(2;\hat{z}_{r+1}^K)] + K(K+1)q_2\hat{z}_m^K +$$

$$\frac{\hat{z}_{r+1}f(\hat{z}_{r+1})[K\hat{z}_{r+1}^K - (K+1)q_1]}{q_1} \qquad (13)$$

$$k_{12} = \frac{v_{12}\theta}{n} = \lim \text{pr } E\left[-\left(\frac{\theta}{n}\right)\left(\frac{\partial^2 L}{\partial\theta\partial K}\right)\right]$$

$$= p - [\Gamma'(2;\hat{z}_m^K) - \Gamma'(2;\hat{z}_{r+1}^K)] - [\Gamma(2;\hat{z}_m^K) -$$

$$\Gamma(2;\hat{z}_{r+1}^K)] - q_2\hat{z}_m^K(K \ln \hat{z}_m + 1) - \frac{\hat{z}_{r+1}f(\hat{z}_{r+1})}{Kq_1}$$

$$\{K\hat{z}_{r+1}^K \ln \hat{z}_{r+1} - (K \ln \hat{z}_{r+1} + 1)q_1\} \qquad (14)$$

$$k_{22} = \frac{v_{22}}{n} = \lim \text{pr } E\left[-\left(\frac{1}{n}\right)\left(\frac{\partial^2 L}{\partial^2 K}\right)\right]$$

$$= \frac{p}{K^2} + \frac{[\Gamma''(2;\hat{z}_m^K) - \Gamma''(2;\hat{z}_{r+1}^K)]}{K^2} + q_2\hat{z}_m^K \ln^2\hat{z}_m +$$

$$\frac{\hat{z}_{r+1}f(\hat{z}_{r+1})}{Kq_1}\{\ln^2 \hat{z}_{r+1}[\hat{z}_{r+1}^K - q_1]\} \qquad (15)$$

where $q_1 = r/n$, $q_2 = (n-m)/n$, and $p = 1 - q_1 - q_2 = (m-r)/n$. As $n \to \infty$ with q_1 and q_2 fixed, z_{r+1} converges in probability to $\hat{z}_{r+1}$ where $F(\hat{z}_{r+1}) = \int_0^{\hat{z}_{r+1}} f(t)dt = q_1$ and z_m converges in probability to $\hat{z}_m$ where $1 - F(\hat{z}_m) = \int_{z_m}^{\infty} f(t)dt = q_2$. The primes indicate differentiation and $\Gamma(a;b)$ is the incomplete Gamma function (Ref 7:559).

Then the information matrix, in terms of the elements k_{ij}, becomes

$$\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{k_{11}n}{\theta^2} & \dfrac{k_{12}n}{\theta} \\ \dfrac{k_{21}n}{\theta} & nk_{22} \end{bmatrix} \tag{16}$$

Let a_{ij} be the elements of the matrix A, which is the inverse of B, then

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{-1} = [B]^{-1} \tag{17}$$

The elements of the variance-covariance matrix are then

$$\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{a_{11}\theta^2}{n} & \dfrac{a_{12}\theta}{n} \\ \dfrac{a_{21}\theta}{n} & \dfrac{a_{22}}{n} \end{bmatrix} \tag{18}$$

since

$$\begin{bmatrix} \dfrac{k_{11}n}{\theta^2} & \dfrac{k_{12}n}{\theta} \\ \dfrac{k_{21}n}{\theta} & k_{22}n \end{bmatrix} \begin{bmatrix} \dfrac{a_{11}\theta^2}{n} & \dfrac{a_{12}\theta}{n} \\ \dfrac{a_{21}\theta}{n} & \dfrac{a_{22}}{n} \end{bmatrix} =$$

$$\begin{bmatrix} k_{11}a_{11} + k_{12}a_{21} & \frac{1}{\theta}(k_{11}a_{12} + k_{12}a_{22}) \\ \theta(k_{21}a_{11} + k_{22}a_{21}) & k_{21}a_{12} + k_{21}a_{22} \end{bmatrix} = I \tag{19}$$

The program given in Appendix C finds the elements of the asymptotic variance-covariance matrix by solving Eqs (13),

(14), and (15) to find the elements k_{ij}, inverting them and converting these to the elements of the variance-covariance matrix by the above transformations.

The variance-covariance matrix is solved by this program for non-integer values of the shape and scale parameters since with the location parameter known the equations are regular for any non-integer value of the parameter estimates. A flowchart and more detailed explanation of the equations in the program are also included in Appendix C.

Calculating System Confidence Limits

The sample values of system reliability, after having been generated, are ordered and thus yield the sample cumulative distribution of the system reliability. From this distribution, the confidence interval and limits can be approximated at any level of confidence. For example, in an ordered sample of 99, the tenth value represents the lower limit of a one sided confidence interval at the 90% confidence level since the a priori probability is .90 that a new random value exceeds the tenth order statistic of a sample of 99. Thus, it is assured that the true reliability is equal to or greater than this value with a confidence of .90.

Verifying the Level of Confidence

In order to insure that the confidence level is accurate, a program was written to generate sample times to failure from the Weibull distribution. Parameters were assumed for each of the component distributions and various size samples

of time to failure were generated for each of the components. This simulates bench testing each of the components to obtain a sample of failure times for each component. The computer program which does this is given, with an explanation of the procedure, in Appendix D.

The true reliability of the system is now known, since the component distribution parameters are known.

The failure times for each component were fed into the Monte Carlo program and a number of sample values of the confidence interval at the required level were generated. The true reliability should be covered by a percentage of the sample confidence intervals equal to the confidence level. For example, if 10 one sided 50% confidence intervals are generated and the true system reliability is .90, a graph of the intervals might appear as shown in Fig. 1 on the next page.

In this case half the intervals cover the true reliability verifying the 50% confidence level (Ref 13:253).

For this paper confidence intervals were generated with each interval calculated from a sample (N) of 99, 499, 999, and 2999 system reliabilities. The results are given in Chapter III.

Systems Analyzed

Two systems are analyzed in this paper. The first system consists of 8 components connected as shown in Fig. 2 for which the required mission time is 75 hours.

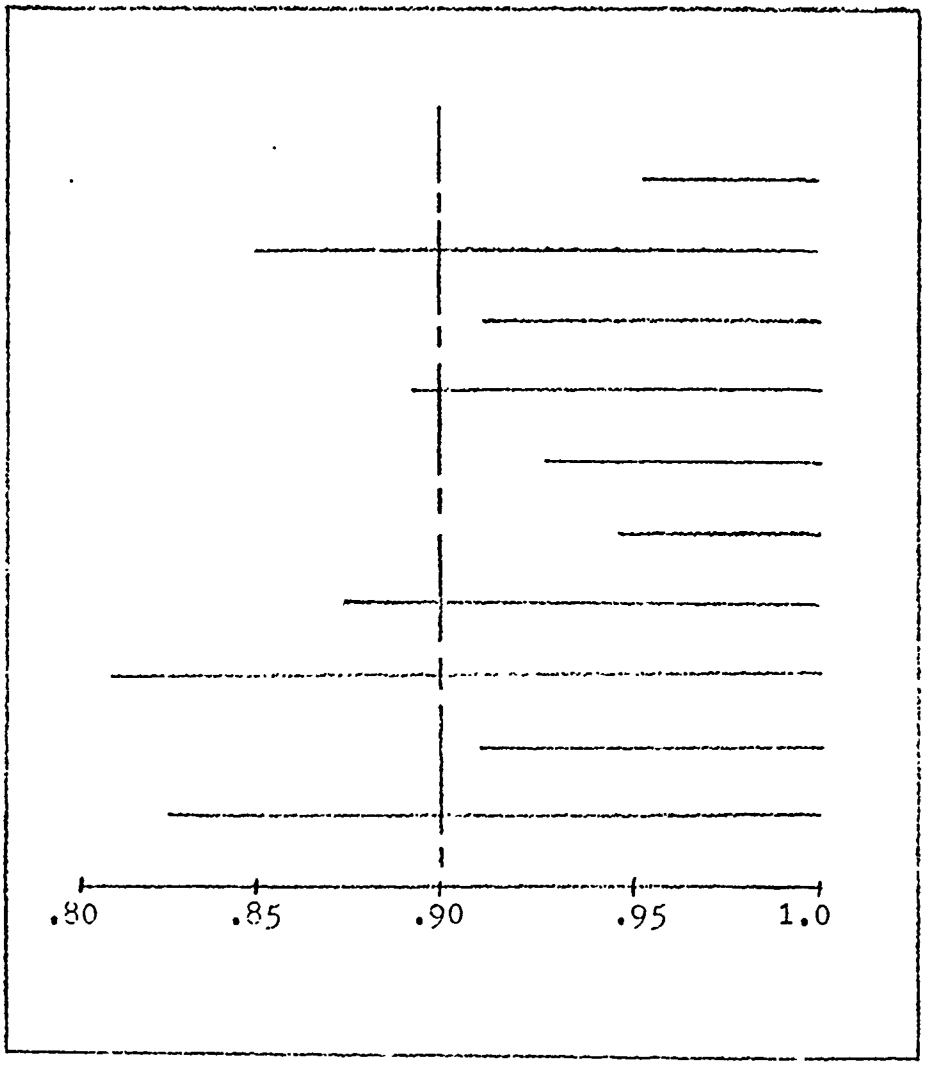

Fig. 1. A Sample of Ten Confidence Intervals Over a True Reliability of .90

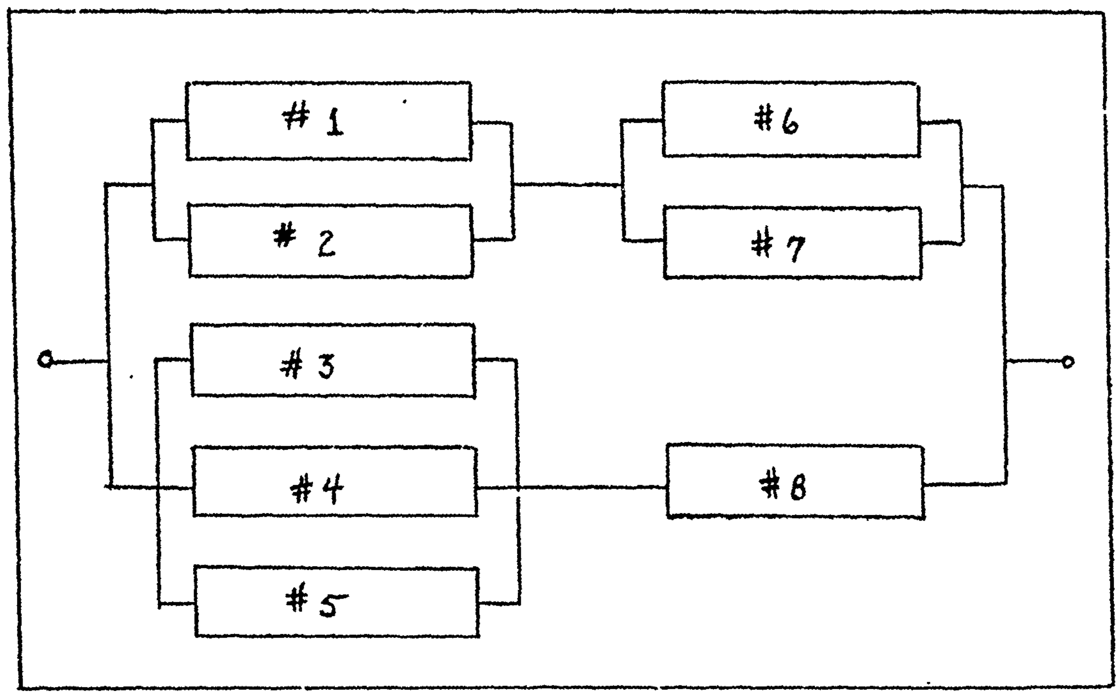

Fig. 2. Schematic of System A.

The components have the following characteristics:

Component #1.

Failure distribution: Exponential

Failure test sample size: 100

Sample censoring: top 6 and bottom 6 values

Location parameter: 0.0

Component #2.

Failure distribution: Exponential

Failure test sample size: 75

Sample censoring: None

Location parameter: 0.0

Component #3.

Failure distribution: Weibull

Failure test sample size: 150

Sample censoring: None

Location parameter: 0.0

Component #4.

Failure distribution: Weibull

Failure test sample size: 200

Sample censoring: None

Location parameter: 0.0

Component #5.

Failure distribution: Exponential

Failure test sample size: 75

Sample censoring: None

Location parameter: 0.0

Component #6.

Failure distribution: Weibull

Failure test sample size: 100

Sample censoring: None

Location parameter: 0.0

Component #7.

Failure distribution: Weibull

Failure test sample size: 200

Sample censoring: None

Location parameter: 0.0

Component #8.

Failure distribution: Weibull

Failure test sample size: 250

Sample censoring: top 12 and bottom 12 values

Location parameter: 0.0

The expression for the reliability of this system is

$$R_s(t) = 1 - Q_a(t)Q_b(t) \tag{20}$$

where $Q_i = 1 - R_i(t)$

$$R_a(t) = [1 - Q_1(t)Q_2(t)][1 - Q_6(t)Q_7(t)]$$

$$R_b(t) = [1 - Q_3(t)Q_4(t)Q_5(t)]R_8(t)$$

The second system (system B) has the configuration shown in Fig. 3 with a required mission time of 100 hours.

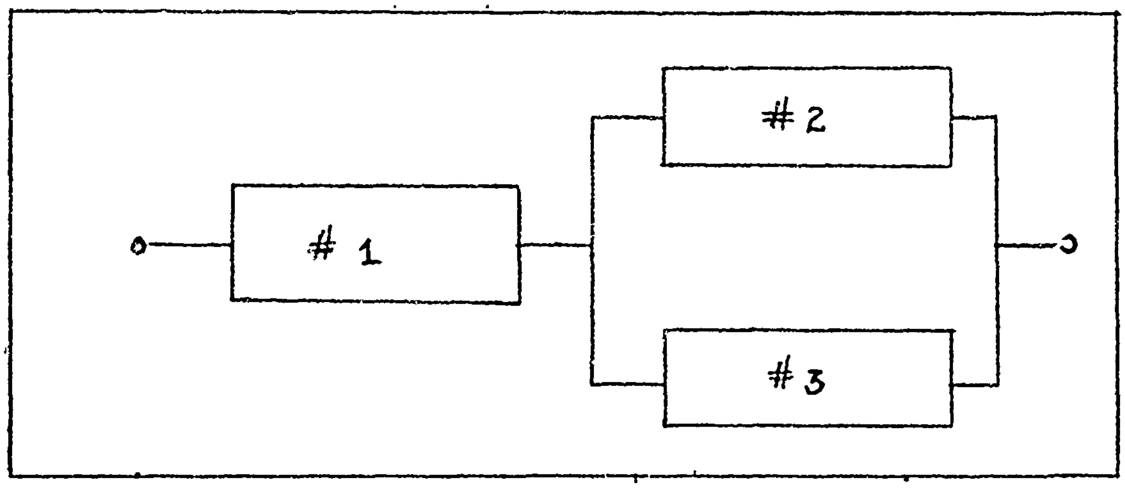

Fig. 3. Schematic of System B.

The components of system B have the following characteristics:

Component #1.

Failure distribution: Weibull

Failure test sample size: 10

Sample censoring: None

Location parameter: 0.0

Component #2.

Failure distribution: Weibull

Failure test sample size: 20

Sample censoring: None

Location parameter: 0.23

Component #3.

Failure distribution: Exponential

Failure test sample size: 30

Sample censoring: None

Location parameter: 0.0

The expression for system reliability for system B, in terms of the i^{th} component reliability is:

$$R_s(t) = R_1(t)(1 - Q_2(t)Q_3(t)) \tag{21}$$

System B was designed to put a strain on the proposed Monte Carlo method due to the small failure test sample size for each of the components.

III. Results

Confidence Limits

It is known that the error associated with a Monte Carlo calculation is proportional to $1/\sqrt{N}$ where N is the number of simulations (Ref 15:259). In this case, the error is statistical, that is, the probable error is proportional to $1/\sqrt{N}$ or the probability is high that the approximate solution does not deviate from the true solution by more than a certain amount (Ref 12:255). The use of a high speed digital computer should decrease the amount of error in the calculation but the machine can add random errors of its own, such as round-off error (Ref 9:11,12). As the problem becomes more complex, the error can be estimated only from the results of the computation.

Tables I and II, on the following page, show the confidence limits determined from simulations of various size for systems A and B. As the simulation size increases the accuracy of the confidence limits at a given confidence level increases and more closely approximates the true confidence limits. Also given are the central processing times for the CDC 6600 computer for each size simulation. No trend should be noted from the bounds given for each size simulation in these tables since each simulation is a random sample of system reliabilities for a single run. The only proper statement that can be made is that the confidence limits for a simulation of size 2999 are more accurate than for one of

Table I
One Sided Confidence Limits
System A

% Conf Level	N= 99		N=499		N=999		N=2999	
	(a)*	(b)**	(a)	(b)	(a)	(b)	(a)	(b)
99	1	.9739	5	.9731	20	.9744	30	.9735
98	2	.9755	10	.9740	30	.9754	60	.9746
97	3	.9760	15	.9748	40	.9761	90	.9754
96	4	.9762	20	.9756	50	.9764	120	.9761
95	5	.9762	25	.9762	60	.9768	150	.9763
90	10	.9772	50	.9775	100	.9779	300	.9777
85	15	.9779	75	.9789	150	.9789	450	.9786
80	20	.9786	100	.9796	200	.9798	600	.9794
70	30	.9801	150	.9807	300	.9808	900	.9806
60	40	.9810	200	.9815	400	.9816	1200	.9815
50	50	.9816	250	.9823	500	.9824	1500	.9823
CP Time	19.4 Sec		23.0 Sec		29.5 Sec		89.6 Sec	

* Order number of the sample
** Confidence limit on system reliability

Table II
One Sided Confidence Limits
System B

% Conf Level	N=99		N=499		N=999		N=2999	
	(a)*	(b)**	(a)	(b)	(a)	(b)	(a)	(b)
99	1	.8788	5	.8680	20	.8427	30	.8287
98	2	.9077	10	.8888	30	.8632	60	.8659
97	3	.9094	15	.8994	40	.8783	90	.8822
96	4	.9125	20	.9046	50	.8864	120	.8919
95	5	.9134	25	.9077	60	.8961	150	.9014
90	10	.9230	50	.9224	100	.9213	300	.9223
85	15	.9357	75	.9324	150	.9334	450	.9339
80	20	.9388	100	.9395	200	.9428	600	.9412
70	30	.9487	150	.9489	300	.9524	900	.9513
60	40	.9549	200	.9558	400	.9576	1200	.9575
50	50	.9616	250	.9605	500	.9626	1500	.9628
CP Time	13.2 Sec.		15.4 Sec.		22.3 Sec.		76.0 Sec.	

* Order number of sample
** Confidence limit on system reliability

size 99. Only the single sided confidence limits are given in these tables because they are the numbers most needed in evaluating the reliability of a new system.

There is also an error associated with the size of the sample of failure times for each component. With the location parameter known, the sample variances and covariances of the asymptotic distribution of the shape and scale parameters are different from the variances and covariances of the small-sample non-asymptotic distribution of the parameters. Analyzing this error is difficult but Harter and Moore determined the error to be relatively small if the location parameter is known, even for samples of failure times as small as size 50 (Ref 7:562, 563).

Lutton compared the asymptotic distribution technique with a double Monte Carlo simulation developed by Moore to approximate the system reliability confidence interval. Moore's method does not use the asymptotic distribution of the parameters but uses a method that has an accuracy that is dependent upon the number of simulations made. Lutton found good agreement between the asymptotic method and the double Monte Carlo method (Ref 11:21-26).

Verification of Confidence Level

In order to check the accuracy of the confidence level, the proposed method was repeated several times. Here, a single repetition of the method is called a run. At first, because of the computer time needed for a single run for

system B, 100 runs were made on both systems A and B for a sample of 99 system reliabilities (simulation size 99). In each run the following is accomplished:

1. Sample failure times for each component are generated from component Weibull distributions with assumed parameters.
2. An ordered sample of 99 system reliabilities is calculated by the proposed method.
3. A counter is increased by one if the true reliability of the system is in the generated confidence interval at any given confidence level.

After all the runs are completed, the value of the counter is divided by the total number of runs. This gives the percentage of runs in which the true reliability is in the confidence interval at a given level; as in the example shown in Fig. 1. The results of this procedure are given in Table III on the next page. As shown in this table the confidence level is found to be reasonably accurate for a simulation size 99, which is the least accurate interval because of the error associated with that size sample.

Table III indicates that the established or required confidence levels in the proposed Monte Carlo method are sometimes optimistic, at other times pessimistic. It must be remembered that these confidence levels were checked by using only 100 runs on a simulation size 99 and it can be expected that the calculated levels would approach the required levels if more runs were possible. Even with these limitations,

Table III
Calculated Confidence Levels
N=99*; 100 Runs

Required Conf. level (%)	Calculated (%)	
	System A	System B
99S**	99	99
98T#	98	95
90S	85	94
90T	87	88
80S	75	86
80T	76	86
70S	62	76
70T	64	79
60S	56	57
60T	59	72
50S	46	45
50T	46	62

* 99 system reliabilities simulated
** One sided confidence interval
# Two sided confidence interval

Table III indicated the accuracy of the proposed method is fairly good at a given confidence level.

To determine the effect of more runs on the calculated confidence level, System B confidence levels were calculated for 300 and 700 runs. This was done only for System B because computer time for each run is not excessive for this three-component system. The results are given in Table IV on the next page. In most cases, if the calculated level was below the required level, the calculated level approaches the required level and then surpasses it when more runs are made. If the calculated confidence level was above the required level, the calculated level approaches the established or required level when more runs are made. In almost all cases the required confidence levels for this system tend to be conservative. This confirms Lutton's results, in which he found the confidence intervals for the asymptotic distribution of the parameters tended to be wider or more conservative than in Moore's double Monte Carlo method. In other words, if the results in Table IV apply to any system analyzed, the confidence limits in the proposed Monte Carlo method tend (in a probabilistic sense) to be slightly conservative for both one and two sided confidence intervals. This is a desirable characteristic from a practical standpoint.

Table IV
System B Confidence Levels with Runs Increased
N=99*

Required Conf. Level (%)	Calculated (%)		
	100 Runs	300 Runs	700 Runs
99S**	99.0	99.0	99.6
98T #	95.0	96.0	97.3
90S	94.0	92.3	92.6
90T	88.0	89.0	90.4
80S	86.0	80.6	82.6
80T	86.0	82.3	83.6
70S	76.0	72.0	73.0
70T	79.0	72.3	74.1
60S	57.0	57.0	60.4
60T	72.0	63.3	64.4
50S	45.0	45.0	49.0
50T	62.0	54.6	53.9

* 99 system reliabilities simulated
** One sided confidence interval
# Two sided confidence interval

IV. Conclusions and Recommendations

Conclusions

The method developed is extremely flexible and gives good accuracy for a wide number of system configurations. The only inputs needed to start the process are the ordered censored or uncensored samples of component failure times and, along with the known value of the component location parameters, any kind of estimates of the component scale and shape parameters. In the latter case a guess will suffice since these estimates are used only to start the method of false position. An increasing or decreasing failure rate may be indicated by the failure times, thus supplying a guess for the shape parameter. Any number within the range of failure times may be used to guess the scale parameter.

Bayesian methods or Boolean algebra may be used to express system reliability in terms of the component reliabilities. In most cases this can be done with a single equation which is inserted in the main program. The proposed method can be applied to any component combination and uses relatively little computer time to generate a large sample from which confidence limits are determined.

The accuracy of the method depends mainly on the number of simulations of system reliability. It also depends, to a lesser extent, on the size of the sample of component failure times. As the size of either of these increases so does the accuracy of the confidence interval. The confidence limits

are easily recovered from the ordered sample of system reliabilities printed as output from the computer program.

Recommendations

It is recommended that further work be done on the proposed Monte Carlo method in the following areas:

Types of Distributions. The method should be extended to include other distributions such as the Logistic and Gamma distributions. Iterative procedures have been developed to obtain the MLE for the parameters of these distributions and could be incorporated into a program such as the one in this paper.

Error Analysis. It is also recommended that a wider variety of component configurations be analyzed to try to determine if there is a direct relationship between the number and configuration of the components and the error. In other words, is error prediction possible, or can error be determined only by analysis of the results?

Analysis of systems with component failure time samples of less than ten would also be useful. If the proposed method remains accurate for such limited component testing, it could be used in systems which have components with a long mean time to failure or where the components are so expensive that only a limited number could be tested to destruction.

Confidence level verification as given in Table III uses a limited number of computer runs on simulation size of 99 system reliabilities. It is recommended that further examination of the accuracy of the confidence level for simulations

of 499, 999, and 2999 also be undertaken to check the increase in accuracy of the confidence interval for these larger size simulations.

Bibliography

1. Bernhoff, Albert O. Confidence Limits for System Reliability Based on Component Test Data. Unpublished thesis. Wright-Patterson Air Force Base, Ohio: Air Force Institute of Technology, August 1963.

2. Buslenko, N. P., et al. The Monte Carlo Method, edited by Yu. A. Shrieder. London: Pergamon Press Limited, 1966.

3. Clarke, Bruce A. and R. L. Disney. Probability and Random Processes for Engineers and Scientists. New York: John Wiley and Sons, 1970.

4. Davis, Philip J. "Gamma Function and Related Functions." in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, edited by Milton Abramowitz and Irene A. Stegun. Washington: National Bureau of Standards, Applied Mathematics Series 55., 1965.

5. Gryna, Frank M., et al. Reliability Training Text (second edition). New York: The Institute of Radio Engineers, Inc., 1960.

6. Harter, H. Leon. "Series Expansion for the Incomplete Gamma Function and its Derivatives." Blanch Anniversary Volume, Wright-Patterson Air Force Base, Ohio: Aerospace Research Laboratories, 1967.

7. Harter, H. Leon and Albert H. Moore. "Asymptotic Variance and Covariances of Maximum-Likelihood Estimators, from Censored Samples, of the Parameters of Weibull and Gamma Populations." Annals of Mathematical Statistics 38, 2: 557-570 (April 1967).

8. -----. "Maximum-Likelihood Estimation of the Parameters of Gamma and Weibull Populations from Complete and from Censored Samples." Technometrics, 7:639-643 (November 1965).

9. Hoel, Paul G. Introduction to Mathematical Statistics (third edition). New York: John Wiley and Sons, 1962.

10. Levy, Louis L. and Albert H. Moore. "A Monte Carlo Technique for Obtaining System Reliability Confidence Limits from Component Test Data." IEEE Transactions on Reliability, R-16, 2:69-72 (September 1967).

11. Lutton, Stephen C. A Monte Carlo Technique for Approximating System Reliability Confidence Limits from Component Test Data. Unpublished thesis. Wright-Patterson Air Force Base, Ohio: Air Force Institute of Technology, December 1967.

12. Milne, William E. Numerical Solutions of Differential Equations. New York: John Wiley and Sons, 1953.

13. Mood, Alexander M. and Franklin A. Graybill. Introduction to the Theory of Statistics. New York: McGraw-Hill Book Co., Inc., 1963.

14. Moore, Albert H. (Personal Communcation) Wright-Patterson Air Force Base, Ohio: Air Force Institute of Technology, 1972.

15. Shooman, Martin L. Probabilistic Reliability: An Engineering Approach. New York: McGraw-Hill Book Co., Inc., 1968.

Appendix A

The Parameter Estimation Subroutine

```
      SUBROUTINE PARES(N,M,C,THETA,EK,MR,STH,SEK)
C     INPUT
C     N=SAMPLE SIZE (BEFORE CENSORING),N=100 OR LESS AS
C              DIMENSIONED
C     SS1=0 IF SCALE PARAMETER THETA IS KNOWN
C     SS1=1 IF SCALE PARAMETER THETA IS TO BE ESTIMATED
C     SS2=0 IF SHAPE PARAMETER K IS KNOWN
C     SS2=1 IF SHAPE PARAMETER K IS TO BE ESTIMATED
C     SS3=0 IF LOCATION PARAMETER C IS KNOWN
C     SS3=1 IF LOCATION PARAMETER C IS TO BE ESTIMATED
C     T(I)=I-TH ORDER STATISTIC OF SAMPLE (I=1,N)
C     M=NUMBER OF OBSERVATIONS REMAINING AFTER CENSORING N-M
C                FROM ABOVE
C     C(1)=INITIAL ESTIMATE (OR KNOWN VALUE) OF C
C     THETA(1)=INITIAL ESTIMATE (OR KNOWN VALUE) OF THETA
C     EK(1)=INITIAL ESTIMATE (OR KNOWN VALUE) OF K
C     MR=NUMBER OF OBSERVATIONS CENSORED FROM BELOW
C     OUTPUT
C     N,SS1,SS2,SS3,M,C(1),THETA(1),EK(1),MR
C             --SAME AS FOR INPUT
C     C(J)=ESTIMATE AFTER J-1 ITERATIONS
C             (OR KNOWN VALUE) OF C
C     THETA(J)=ESTIMATE AFTER J-1 ITERATIONS
C             (OR KNOWN VALUE) OF THETA
C     EK(J)=ESTIMATE AFTER J-1 ITERATIONS
C           (OR KNOWN VALUE) OF K
C       (MAXIMUM VALUE OF J AS PRESENTLY DIMENSIONED IS 500)
C     EL=NATURAL LOG. OF LIKELIHOOD FOR C(J),THETA(J),EK(J)
      DIMENSION T(500),C(500),THETA(500),EK(500),Y(56),Y(55)
      SS1=1.
      SS2=1.
      SS3=1.
      IF(N)55,56,104
  104 FN=N
      READ 3,(T(I),I=1,N)
    3 FORMAT(5F16.5)
      IF(M)55,56,110
  110 PRINT 115
  115 FORMAT(1H ,/,1H ,10HESTIMATES:)
      PRINT 120
  120 FORMAT(1H ,9HLOCATION,7X,5HSCALE,5X,5HSHAPE)
      PRINT 123,C(1),THETA(1),EK(1)
  123 FORMAT(1H ,3E8.5)
      PRINT 125
  125 FORMAT(1H ,//,1H ,10HITERATIONS)
      EM=M
   31 ELNM=0.
      EMR=MR
      MRP=MR+1
   33 NM=N-M+1
      DO 34 I=NM,N
      FI=I
   34 ELNM=ELNM+ALOG(FI)
      IF (MR) 56,75,74
   74 DO 75 I=1,MR
      FI=I
   75 ELNM=ELNM-ALOG(FI)
```

```
 35     DO 70 J=1,NS1
        IF (J-1) 36,20,37
 37     JJ=J-1
        SK=0.
        SL=0.
        DO 6 I=MPE,M
 6      SK=SK+(T(I)-C(JJ))**EK(JJ)
        IF (SS1) 7,7,8
 7      THETA(J)=THETA(JJ)
        GO TO 9
 8      IF (MP) 66,13,?
 13     THETA(J)=((SK+(EN-EM)*(T(M)-C(JJ))**EK(JJ))/EM)
       C**(1./EK(JJ))
        GO TO 9
 20     X(1)=THETA(JJ)
        LS=
        DO 21 L=1,50
        LL=L-1
        LP=L+1
        X(LP)=Y(L)
        ZRK=((T(MxP)-C(JJ))/X(L))**EK(JJ)
        Y(L)=-EK(JJ)*(EN-EM )/Y(L)+EK(JJ)*SK/Y(L)**(EK(JJ)+1.)
       C+EK(JJ)*(EN-EM)*(T(M)-C(JJ))**EK(JJ)/X(L)**(EK(JJ)+1.
       C)-EMP*EK(JJ)*ZRK*EXP(-ZRK)/(X(L)*(1.-EXP(-ZRK)))
        IF (Y(L)) 53,73,54
 53     LS=LS-1
        IF (LS+L) 58,55,58
 54     LS=LS+1
        IF (LS-L) 58,56,58
 55     X(LP)=.5*Y(L)
        GO TO 21
 56     Y(LP)=1.5*Y(L)
        GO TO 21
 58     IF (Y(L)*Y(LL)) 60,73,59
 59     LL=LL-1
        GO TO 58
 60     X(LP)=X(L)+Y(L)*(X(L)-Y(LL))/(Y(LL)-Y(L))
 61     IF (ABS(X(LP)-Y(L))-1.E-6) 73,73,21
 21     CONTINUE
 73     THETA(J)-Y(LP)
 9      EK(J)=EK(JJ)
 10     IF (SS2) 12,12,11
 11     DO 17 I=MPE,M
 17     SL=SL+ALOG(T(I)-C(JJ))
        X(1)=EK(J)
        LS=
        DO 51 L=1,50
        SLK=0.
        DO 18 I=MPE,M
 18     SLK=SLK+(ALOG(T(I)-C(JJ))-ALOG(THETA(J)))*(T(I)-C(JJ))
       C**X(L)
        LL=L-1
        LP=L+1
        X(LP)=Y(L)
        ZRK=((T(MPE)-C(JJ))/THETA(J))**Y(L)
        Y(L)=(EM-EMR)*(1./Y(L)-ALOG(THETA(J)))+SL-SLK/THETA(J)
       C**Y(L)+(EN-EM)*(ALOG(THETA(J))-ALOG(T(M)-C(JJ)))*(T(M)
       C-C(JJ))**Y(L)/THETA(J)**X(L)+EMP*ZRK*(ALOG(ZRK)/Y(L))
       C*EXP(-ZRK)/(1.-EXP(-ZRK))
        IF (Y(L)) 43,53,44
 43     LS=LS-1
        IF (LS+L) 47,45,47
 44     LS=LS+1
        IF (LS-L) 47,46,47
 45     X(LP)=.5*Y(L)
        GO TO 51
 46     Y(LP)=1.5*Y(L)
        GO TO 51
 47     IF (Y(L)*Y(LL)) 49,53,48
 48     LL=LL-1
        GO TO 47
```

```
49     X(LP)=X(L)+Y(L)*(X(L)-X(LL))/(Y(LL)-Y(L))
50     IF (ABS(X(LP)-X(L))-1.E-4) 52,52,51
51     CONTINUE
52     EK(J)=X(LP)
12     C(J)=C(JJ)
62     IF (SS3) 25,25,14
14     IF (1.-EK(J)) 1 ,78,78
78     IF (SS1+SS2) 57,57,16
16     X(1)=C(J)
       LS=0
       DO 23 L=1,55
       SK1=0.
       SR=0.
       DO 15 I=MPP,M
       SK1=SK1+(T(I)-X(L))**(EK(J)-1.)
15     SR=SR+1./(T(I)-X(L))
       LL=L-1
       LP=L+1
       X(LP)=X(L)
       ZPK=((T(MPP)-X(L))/THETA(J))**EK(J)
       Y(L)=(1.-EK(J))*SR+EK(J)*(SK1+(N-MP)*(T(M)-X(L))
      C**(EK(J)-1.))/THETA(J)**EK(J)-MPR*EK(J)*ZPK*EXP(-ZPK)
      C/((T(MPP)-X(L))*(1.-EXP(-ZPK)))
       IF (Y(L)) 39,24,40
39     LS=LS-1
       IF (LS+L) 70,41,70
40     LS=LS+1
       IF (LS-L) 70,42,70
41     X(LP)=.5*X(L)
       GO TO 23
42     X(LP)=.5*X(L)+.5*T(1)
       GO TO 23
70     IF (Y(L)*Y(LL)) 72,24,71
71     LL=LL-1
       GO TO 70
72     X(LP)=X(L)+Y(L)*(X(L)-X(LL))/(Y(LL)-Y(L))
22     IF (ABS(X(LP)-X(L))-1.E-4) 24,24,23
23     CONTINUE
24     C(J)=X(LP)
       GO TO 25
57     C(J)=T(1)
25     IF (MP) 38,38,68
38     DO 63 I=1,M
       IF (C(J)+1.E-4-T(I)) 68,67,67
67     MP=MP+1
63     C(1)=T(1)
68     IF (MP) 36,69,71
69     SK=0.
       SL=0.
       DO 36 I=MPP,M
       SK=SK+(T(I)-C(J))**EK(J)
36     SL=SL+ALOG(T(I)-C(J))
       ZPK=((T(MPP)-C(J))/THETA(J))**EK(J)
       EL=ELI+(N-MPR)*(ALOG(EK(J))-EK(J)*ALOG(THETA(J)))+
      C(EK(J)-1.)*SL-(SK+(N-M)*(T(M)-C(J))**EK(J))/(THETA
      C(J)**EK(J))+MPR*ALOG(1.-EXP(-ZPK))
       PRINT 26,C(J),THETA(J),EK(J),EL
  26   FORMAT(1H ,4X,4E15.6)
       IF (J-1) 7 ,27,27
27     IF (ABS(C(J)-C(JJ))-1.E-4) 28,28,30
28     IF (ABS(THETA(J)-THETA(JJ))-1.E-4) 29,29,30
  29   IF(ABS(EK(J)-EK(JJ))-1.E-4)126,126,70
30     CONTINUE
 126   PRINT 130
 130   FORMAT(1H ,/,1H ,19HFINAL ESTIMATE IS:)
       PRINT 127
 127   FORMAT(1H ,4X,8HLOCATION,6X,5HSCALE,8X,5HSHAPE,6X,
      C13HLOG OF LT.)
       PRINT 26,C(J),THETA(J),EK(J),EL
```

```
      PTH=THETA(J)
      PEK=EK(J)
      GO TO 140
   66 PRINT 135
  135 FORMAT(1H ,20HALL SAMPLES CENSORED,/)
      PEK=0.
      PTH=0.
  140 CONTINUE
      RETURN
      END
```

Appendix B

Generating the System Reliabilities

The main Fortran program SYSRE generates the system reliabilities. The inputs to this program are the number of components in the system (a maximum of ten as presently dimensioned), the number of simulations desired, and the mission time. Censoring information is supplied to the program along with the known value of the location parameter. The failure times are read for each component when the subroutine PARES is called and the true value of the location parameter is read into the main program for each component. The output from the main program is a sample of ordered system reliabilities from which the confidence limits can be read for a given confidence level. The flowchart is given in Fig. 4 and the function subprogram which is used to generate pseudo-random numbers from the standard normal distribution is explained in the notes.

Notes (refer to Fig. 4):

1. The ordered component failure times are read into subroutine PARES and the individual component MLE's of the shape and scale parameters are calculated and stored each time the subroutine is called.
2. When the covariances are equal to zero, the symmetric matrix to the 1/2 power is simply the square root of the non-zero entries of the 2 X 2 matrix.

3. In this case $[V]^{1/2}$ is evaluated by using the matrix formula

$$[V]^{1/2} = [S]^{-1}[\Lambda]^{n}[S] \tag{22}$$

where $[\Lambda]$ and $[S]$ are matrices that diagonalize $[V]$, and the diagonal elements of $[\Lambda]$ are the eigenvalues of $[V]$ (Ref 3:244-251).

4. To generate samples from the standard normal distribution the central limit theorem was used where the distribution

$$t = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \tag{23}$$

approaches the standard normal distribution as $n \to \infty$. If $t = x - 1/2$ where x is a pseudo-random number from the uniform distribution, it is found that for a sample of 12 numbers

$$Y = \sum_{i=1}^{12} x_i - 6 \tag{24}$$

has a distribution that approaches the standard normal distribution where the x_i are pseudo-random numbers from the uniform distribution (Ref 9:145, 146). Function subprogram RANOR accomplishes the above and returns the numbers to SYSRE.

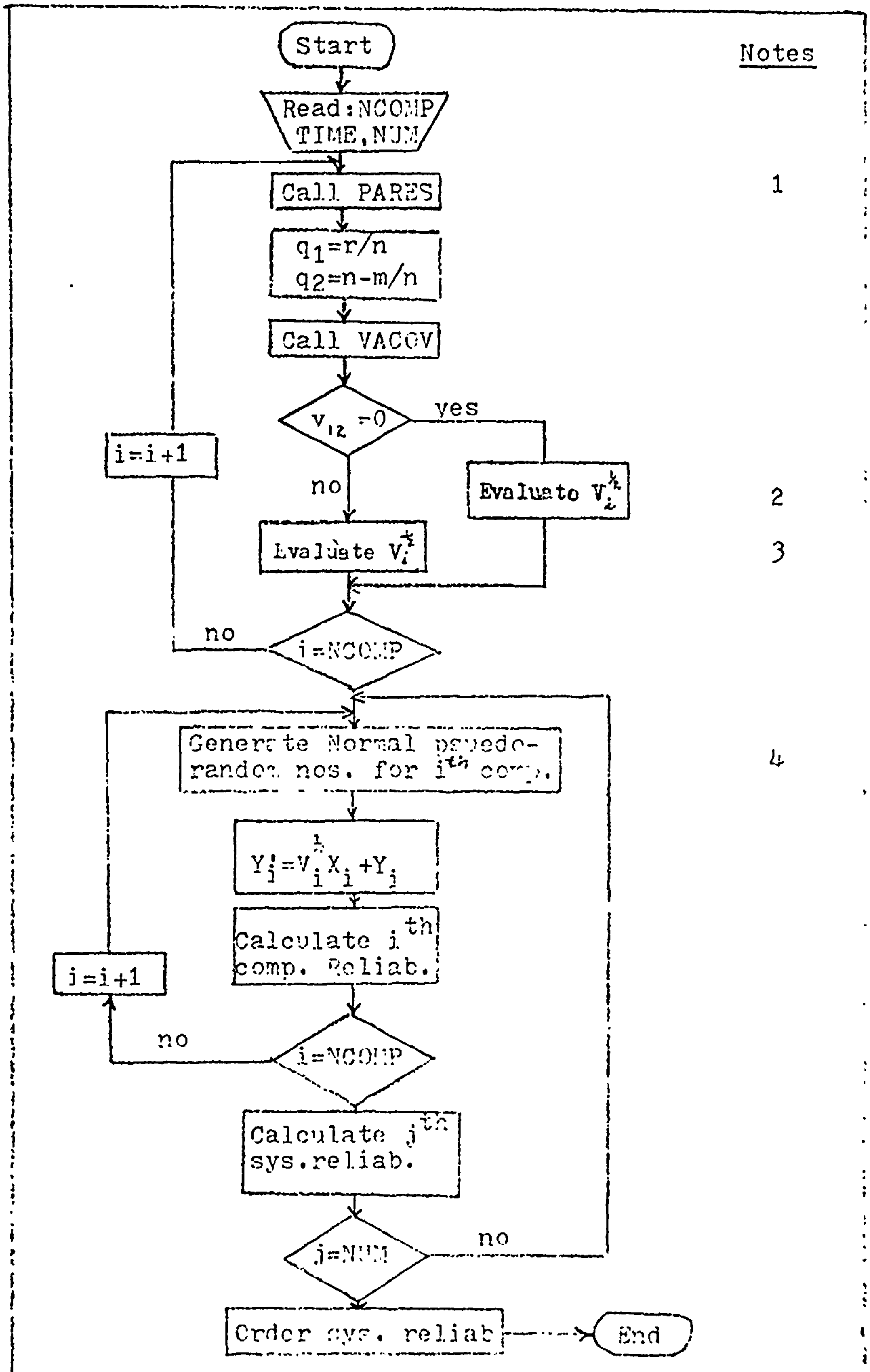

Fig. 4. Flowchart for Main Program

```
      PROGRAM SYSPF(INPUT,OUTPUT)
C*****NCOMP=NUMBER OF COMPONENTS IN SYSTEM
C*****NUM=NUMBER OF SIMULATIONS DESIRED
C*****TIME=MISSION TIME
C*****EEK=ESTIMATE OF SHAPE PARAMETER FOR THAT COMPONENT
C*****ETH=ESTIMATE OF SCALE PARAMETER FOR THAT COMPONENT
C*****XC=LOCATION PARAMETER FOR THAT COMPONENT
C*****NSAMP=SAMPLE SIZE BEFORE CENSORING(N=500 OR LESS)
C*****MABOV=QUANTITY M WHERE N-M SAMPLES CENSORED FROM ABOVE
C*****MBLOW=NUMBER OF SAMPLES CENSORED FROM BELOW
      DIMENSION S(2,2),V(10,2,2),R(50),RS(3000),
     CNSAMP(100),MABOV(100),MBLOW(100),EEK(10),ETH(10),
     CTXC(10),TH(10),EK(10),TC(550),ET(550),EKK(550)
      RS(100)=0.
      RS(500)=0.
      RS(1000)=0.
      RS(3000)=0.
      READ 50,NCOMP,NUM,TIME
   50 FORMAT(I5,I5,F20.8)
      PRINT 55
   55 FORMAT(1H ,17HNO. OF COMPONENTS,5X,8HNO. OF   ,
     C11HSIMULATIONS,5X,12HMISSION TIME)
      PRINT 60,NCOMP,NUM,TIME
   60 FORMAT(1H ,7X,I5,17X,I5,15X,F20.8)
      DO 132 K=1,NCOMP
      PRINT 63,K
   63 FORMAT(1H ,50(*,*),//,1H ,16HCOMPONENT NUMBER,I6)
      READ 70,NSAMP(K),MABOV(K),MBLOW(K),ETH(K),
     CEEK(K),TXC(K)
   70 FORMAT(3I5,3F8.4)
      TC(1)=TXC(K)
      ET(1)=ETH(K)
      EKK(1)=EEK(K)
      CALL PARES(NSAMP(K),MABOV(K),TC,ET,
     CEKK,MBLOW(K),TH(K),EK(K))
      IF(TH(K).EQ.0.)GO TO 500
      AA=MBLOW(K)
      BB=MABOV(K)
      CC=NSAMP(K)
      Q1=AA/CC
      Q2=(CC-BB)/CC
      CALL VACOV(Q1,Q2,EK(K),CC,TH(K),V11,V12,V22)
      A=V11
      B=V12
      C=V22
      IF (V12)110,100,110
  100 V(K,1,1)=SQRT(V11)
      V(K,1,2)=0.
      V(K,2,1)=0.
      V(K,2,2)=SQRT(V22)
      V(1,2)=0.
      V(2,1)=0.
      V(2,2)=SQRT(V22)
      GO TO 112
  110 Y=.5*SQRT(A*A-2.*A*C+C*C+4.*B*B)
      X1=.5*(A+C)+Y
      X2=.5*(A+C)-Y
      R1=SQRT(1.+(X1-A)**2/B**2)
      R2=SQRT(1.+(X2-C)**2/B**2)
      S(1,1)=1./R1
      S(2,2)=1./R2
      S(1,2)=(X2-C)/(B*R2)
      S(2,1)=(X1-A)/(B*R1)
      X1=SQRT(X1)
      X2=SQRT(X2)
      D1=S(1,1)*X1
      D2=S(1,2)*X2
      D3=S(2,1)*X1
      D4=S(2,2)*X2
```

```
      V(K,1,1)=C1*S(1,1)+C2*S(1,2)
      V(K,1,2)=C1*S(2,1)+C2*S(2,2)
      V(K,2,1)=C3*S(1,1)+C4*S(1,2)
      V(K,2,2)=C3*S(2,1)+C4*S(2,2)
  112 PRINT 115
  115 FORMAT(1H0,*SOLUTION OF COV. TO 1/2 POWER*)
      PRINT 130,V(K,1,1),V(K,1,2),V(K,2,1),V(K,2,2)
  130 FORMAT(1H ,4F10.5)
  132 CONTINUE
C*************GENERATION OF SAMPLE VALUE OF PARAMETERS
  133 DO 170 J=1,NUM
      DO 150 K=1,NCOMP
      W=RANOR(X)
      B=RANOR(X)
      ST=V(K,1,1)*W+V(K,1,2)*B
      SB=V(K,2,1)*W+V(K,2,2)*B
      TH1=ST+TH(K)
      EK1=SB+EK(K)
C*****GENERATING SAMPLE WEIBULL COMPONENT RELIABILITIES
      R(K)=EXP(-(((TIME-TYO(K))/TH1)**EK1))
  150 CONTINUE
C*****GENERATING SAMPLE SYSTEM RELIABILITY
      RS(J)=R(1)*(1.-(1.-R(2))*(1.-R(3)))
  170 CONTINUE
C*****ORDERING SAMPLE SYSTEM RELIABILITIES
      ML=1
  175 MH=NUM-1
  176 MHT=MH
      ISW=1
      DO 200 I=ML,MHT
      IF(RS(I)-RS(I+1))200,200,180
  180 GO TO (185,190),ISW
  185 ISW=2
      IF(I-1)190,190,187
  187 ML=I-1
  190 TEMP=RS(I)
      RS(I)=RS(I+1)
      RS(I+1)=TEMP
      MH=I-1
  200 CONTINUE
      GO TO (205,176),ISW
  205 PRINT 204,NUM
  204 FORMAT(1H ,//,1H ,*SYSTEM RELIABILITY SAMPLE, N=*,I4,
     C//)
  206 IF(NUM.NE.99)GO TO 250
  201 DO 202 J=1,25
      J1=J+25
      J2=J+50
      J3=J+75
  202 PRINT 203,J,RS(J),J1,RS(J1),J2,RS(J2),J3,RS(J3)
  203 FORMAT(1H ,4(I5,3X,F7.4))
      GO TO 500
  250 IF(NUM.NE.400)GO TO 300
      DO 270 J=1,100
      J1=J+100
      J2=J+200
      J3=J+300
      J4=J+400
  270 PRINT 275,J,RS(J),J1,RS(J1),J2,RS(J2),J3,RS(J3),J4,
     CRS(J4)
  275 FORMAT(1H ,5(I4,2X,F6.4,2X))
      GO TO 500
  300 IF(NUM.NE.999)GO TO 400
      DO 310 J=1,200
      J1=J+200
      J2=J+400
      J3=J+600
      J4=J+800
```

```
310 PRINT 315,J,RS(J),J1,RS(J1),J2,RS(J2),J3,RS(J3),J4,
   CRS(J4)
315 FORMAT(1H ,5(I4,2X,F5.4))
    GO TO 500
400 IF(NUM.NE.3000)GO TO 500
    DO 410 J=1,500
    J1=J+500
    J2=J+1000
    J3=J+1500
    J4=J+2000
    J5=J+2500
410 PRINT 415,J,RS(J),J1,RS(J1),J2,RS(J2),J3,RS(J3),J4,
   CRS(J4),J5,RS(J5)
415 FORMAT(1H ,6(I5,F6.4))
    GO TO 500
500 CONTINUE
    END


    FUNCTION RANOR(Y)
    DIMENSION X(20)
    J=0
  1 SUM=0.
    DO 10 I=1,12
    X(I)=RANF(Y)
    SUM=SUM+X(I)
 10 CONTINUE
    Y=SUM-6.0
    RANOR=Y
    RETURN
    END
```

Appendix C

The Variance-Covariance Matrix

Subroutine VACOV solves Eqs (13), (14), and (15); inverts the result and multiplies these by the appropriate constants as shown in Eq (17). The function subprograms GAMI, DGAMI, and D2GAMI compute values of the incomplete Gamma function and its derivatives by using a series expansion as shown in Ref 6. The function subprograms GAM, DGAM and D2GAM use the asymptotic expansion formula to calculate values of the complete Gamma function and its derivatives (Ref 4:257). The flowchart for subroutine VACOV is given in Fig. 5.

Notes (see Fig. 5):

1. $$v_{11} = -Kp + K(K+1)[\Gamma(2;\hat{z}_m^K)] + K(K+1)q_2\hat{z}_m^K \tag{25}$$

$$v_{22} = \frac{p}{K^2} + \frac{[\Gamma''(2;\hat{z}_m^K)]}{K^2} + q_2\hat{z}_m^K \ell n^{2} \hat{z}_m \tag{26}$$

$$v_{12} = p - [\Gamma'(2;\hat{z}_m^K)] - [\Gamma(2;\hat{z}_m^K)] - q_2\hat{z}_m^K[1 + K \ell n \hat{z}_m^K] \tag{27}$$

2. $$v_{11} = -Kp + K(K+1) \tag{28}$$

$$v_{22} = \frac{p}{K^2} + \frac{\Gamma''(2.0)}{K^2} \tag{29}$$

$$v_{12} = \psi - \Gamma'(2.0) - 1.0 \tag{30}$$

3. $$v_{11} = v_{11} + [-K(K+1)]\Gamma(2.0;\hat{z}^{K}_{r+1}) + \frac{q_1\hat{z}_{r+1}f(\hat{z}_{r+1})[K\hat{z}^{K}_{r+1} - (K+1)q_1]}{q_1^2} \tag{31}$$

$$v_{22} = v_{22} - \frac{\Gamma''(2.0;\hat{z}^{K}_{r+1})}{K^2} + \frac{q_1\hat{z}_{r+1}f(\hat{z}_{r+1})\ \ell n^2\ (\hat{z}_{r+1})[\hat{z}^{K}_{r+1} - q_1]}{q_1^2 K} \tag{32}$$

$$v_{12} = v_{12} + \Gamma'(2.0;\hat{z}^{K}_{r+1}) + \Gamma(2.0;\hat{z}^{K}_{r+1}) - \frac{q_1\hat{z}_{r+1}f(\hat{z}_{r+1})[K\hat{z}^{K}_{r+1}\ \ell n\ \hat{z}_{r+1} - (K\ \ell n\ \hat{z}_{r+1}+1)q_1]}{q_1^2 K}$$

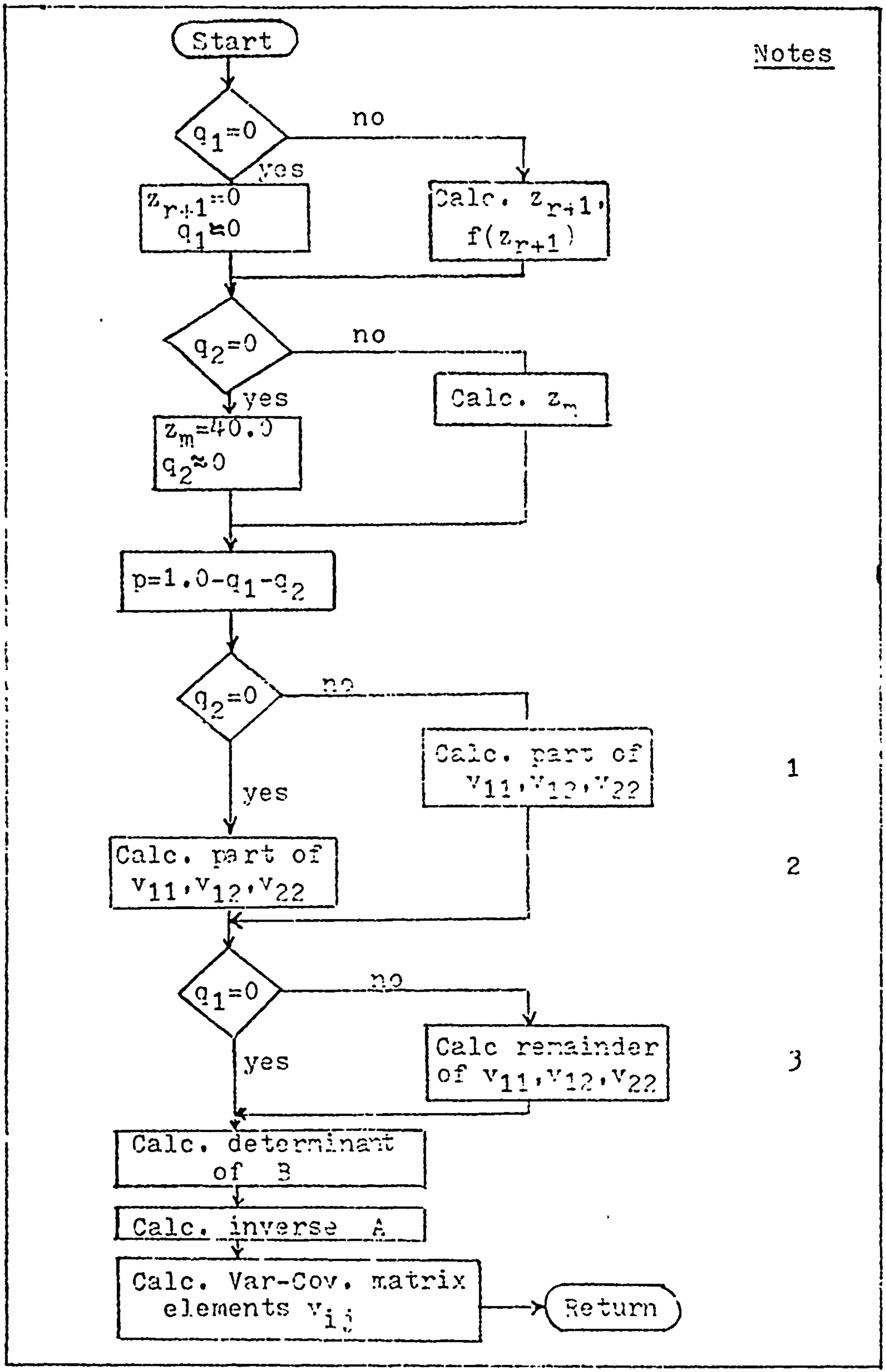

Fig. 5. Flowchart for the Variance-Covariance Subroutine

```
      SUBROUTINE VACOV(Q1,Q2,EK,SNUM,TH,S11,S12,S22)
      K=0
      J=0
      IF(Q1.EQ.1.)GO TO 1
      IF(Q2.EQ.1.)GO TO 1
      GO TO 3
    1 PRI*T 2
    2 FORMAT(1H ,14HALL N CENSORED)
      GO TO 27
    3 CONTINUE
      IF (Q1.EQ.0.)GO TO 4
      ZR=(-ALOG(1.0-Q1))**(1.0/EK)
      FLZR=EK*ZR**(EK-1.0)*EXP(-ZR**EK)
      GO TO 5
    4 ZR=0.0
      K=1
      Q1=1.E-28
    5 IF(Q2.EQ.0.)GO TO 7
      ZM=(-ALOG(Q2))**(1.0/EK)
      GO TO 8
    7 ZM=4.0
      J=1
      Q2=1.E-28
    8 P=1.0-Q1-Q2
      IF(J.EQ.1)GO TO 14
      V11=-EK*P+EK*(EK+1.0)*GAMI(ZM**EK,2.0)+EK*(EK+1.0)
     C *Q2*ZM**EK
      V22=P/EK**2+GAMMT(ZM**EK,3.0)/EK**2+Q2*ZM**EK*ALOG
     C(ZM)**2
      V12=P-GAMMT(ZM**EK,2.0)-GAMI(ZM**EK,2.0)
     C -Q2*ZM**EK*(1.0+EK*ALOG(ZM))
      GO TO 17
   14 V11=-EK*P+EK*(EK+1.0)
      V22=P/EK**2+P...(3.0)/EK**2
      V12=P-G...(2.0)-1.
   17 IF(K.EQ.1)GO TO 24
      V11=V11+(-EK*(EK+1.0)*GAMI(ZR**EK,2.0)+EK*ZR
     C *FLZR*(EK*ZR**EK-(EK+1.0)*Q1)/Q1**2)
      V22=V22-GAMI(ZR**EK,3.0)/EK**2+Q1*ZR-FLZR*ALOG(ZR)**2*
     C (ZR**EK-Q1)/(EK*Q1**2)
      V12=V12+GAMMT(ZR**EK,2.0)+GAMI(ZR**EK,2.0)-Q1
     C *ZR*FLZR*(EK*ZR**EK*ALOG(ZR)-(EK*ALOG(ZR)+1.0)*
     CQ1)/(EK*Q1**2)
   24 D=V11*V22-V12**2
      S11=V22/D
      S12=-V12/D
      S21=S12
      S22=V11/D
      S11=S11*(TH**2)/SNUM
      S12=S12*TH/SNUM
      S22=S22*(1./SNUM)
      PRINT 28
   28 FORMAT(1H ,26HSOLUTION TO VAR-COV MATRIX,//,1H ,
     C6X,3HV11,9X,3HV12,9X,3HV22,/)
      PRINT 26,S11,S12,S22
   26 FORMAT(3E12.5)
   27 CONTINUE
      RETURN
      END
```

```
      FUNCTION GAM(Y)
      Z=Y
      G=0.0
1     IF (Z-9.0) 2,3,3
2     G=G-ALOG(Z)
      Z=Z+1.0
      GO TO 1
3     GAM=G+(Z-.5)*ALOG(Z)-Z+.5*ALOG(2.0*3.1415926535897)
     C+1.0/(12.0*Z)-1.0/(360.0*Z**3)+1.0/(1260.0*Z**5)
     C-1.0/(1680.0*Z**7)+1.0/(1188.0*Z**9)-691.0/
     C(360360.0*Z**11)+1.0/(156.0*Z**13)
      GAM=EXP(GAM)
      RETURN
      END


      FUNCTION DGAM(Y)
      Z=Y
      DG=0.0
1     IF (Z-9.0) 2,3,3
2     DG=DG-1.0/Z
      Z=Z+1.0
      GO TO 1
3     DGAM=DG+(Z-.5)/Z+ALOG(Z)-1.0-1.0/(12.0*Z**2)+1.0
     C/(120.0*Z**4)-1.0/(252.0*Z**6)+1.0/(240.0*Z**8)
     C-1.0/(132.0*Z**10)+691.0/(32760.0*Z**12)
     C-1.0/(12.0*Z**14)
      DGAM=DGAM*GAM(Y)
      RETURN
      END


      FUNCTION D2GAM(Y)
      Z=Y
      D2G=0.0
1     IF (Z-9.0) 2,3,3
2     D2G=D2G+1.0/Z**2
      Z=Z+1.0
      GO TO 1
3     D2GAM=D2G+1.0/Z+1.0/(2.0*Z**2)+1.0/(6.0*Z**3)-
     C1.0/(30.0*Z**5)+1.0/(42.0*Z**7)-1.0/(30.0*Z**9)
     C+5.0/(66.0*Z**11)-691.0/(2730.0*Z**13)+7.0/
     C(6.0*Z**15)
      D2GAM=(D2GAM*GAM(Y)**2+DGAM(Y)**2)/GAM(Y)
      RETURN
      END


      FUNCTION GAMI(W,Z)
      DIMENSION U(50)
      U(1)=W**Z/Z
      SU=U(1)
      DO 1 L=2,50
      LL=L-1
      FLL=LL
      U(L)=(-U(LL)/FLL)*W*(Z+FLL-1.0)/(Z+FLL)
1     SU=SU+U(L)
      GAMI=SU
      RETURN
      END
```

```
      FUNCTION DGAMI(N,Z)
      DIMENSION U(50),V(50)
      U(1)=N**Z*ALOG(N)/Z
      V(1)=N**Z/Z**2
      SU=U(1)-V(1)
      DO 1 L=2,50
      LL=L-1
      FLL=LL
      U(L)=(-U(LL)*N/FLL)*(Z+FLL-1.0)/(Z+FLL)
      V(L)=-V(LL)*N*(Z+FLL-1.0)**2/((Z+FLL)**2*FLL)
    1 SU=SU+U(L)-V(L)
      DGAMI=SU
      RETURN
      END

      FUNCTION D2GAMI(N,Z)
      DIMENSION U(50),V(50),X(50)
      U(1)=N**Z*ALOG(N)**2/Z
      V(1)=2.0*N**Z*ALOG(N)/Z**2
      X(1)=2.0*N**Z/Z**3
      SU=U(1)-V(1)+X(1)
      DO 1 L=2,50
      LL=L-1
      FLL=LL
      U(L)=-U(LL)*N*(Z+FLL-1.0)/((Z+FLL)*FLL)
      V(L)=-V(LL)*N*(Z+FLL-1.0)**2/((Z+FLL)**2*FLL)
      X(L)=-X(LL)*N*(Z+FLL-1.0)**3/((Z+FLL)**3*FLL)
    1 SU=SU+U(L)-V(L)+X(L)
      D2GAMI=SU
      RETURN
      END
```

Appendix D

Generating Sample Component Times to Failure

The program in this appendix was used to generate sample failure times for each component. If the points in the cumulative distribution function are represented by random numbers in the uniform distribution over the interval 0.0 to 1.0 (represented by Y) then

$$Y = F(t) = 1 - \exp\left[-\left(\frac{t-C}{\theta}\right)^{K}\right] \tag{34}$$

for the Weibull distribution. The inverse relation is

$$t = -\theta[-\ell n(1-Y)]^{1/K} + C \tag{35}$$

where t is a random value from the Weibull distribution and Y is a pseudo-random number from the above uniform distribution (Ref 15:258). Program RSAMP calculates the true component and system reliability for the given number of components using the required mission time. It then generates the individual component failure times and orders them so that the order statistics can be read into the subroutine which calculates the MLE for shape and scale parameters. This program was used to check the accuracy of the computed confidence interval and confidence level and is not a part of the method proposed.

```
      PROGRAM RSAMP(INPUT,OUTPUT,PUNCH)
      DIMENSION TEK(50),TTH(50),TXC(50),NSAMP(100),MAROV(100),
     1MRLCW(100),RC(50),T(100,50)
      READ 11,NCOMP,TIME
   11 FORMAT(I2,F8.4)
      PRINT 12,NCOMP,TIME
   12 FORMAT(1H ,21HNUMBER OF COMPONENTS=,I2,13HMISSION TIME=,
     CF8.4,//)
      DO 20 K=1,NCOMP
      READ 15,TEK(K),TTH(K),TXC(K),NSAMP(K),MAROV(K),MRLCW(K)
   15 FORMAT(3F10.5,3I3)
      RC(K)=EXP(-(((TIME-TXC(K))/TTH(K))**TEK(K)))
   20 CONTINUE
C*********ENTER MODEL FOR SYSTEM RELIABILITY HERE***************
      TRS=1.-((1.-RA)*(1.-RB))
      PRINT 25,TRS
   25 FORMAT(1H ,ACTUAL RELIABILITY FOR SYSTEM=,F8.6)
C***********GENERATING SAMPLE VALUES FROM WEIBULL DIST.****
      DO 30 K=1,NCOMP
      NN=NSAMP(K)
      PUNCH 26,NSAMP(K),MAROV(K),MRLCW(K),TTH(K),TEK(K),TXC(K)
    2 FORMAT(3I3,3F10.4)
      CALL RANWI(TEK(K),TTH(K),TXC(K),T(1,K),NSAMP(K),K)
      PUNCH 27,(T(I,K),I=1,NN)
   27 FORMAT(5F12.5)
   30 CONTINUE
      END


      SUBROUTINE RANWI(EK,TH,XC,T,N,K)
      DIMENSION T(50)
      PRINT 2,K
    2 FORMAT(1H ,50(*.*),//,1H ,17HCOMPONENT NUMBER ,I4,//)
      PRINT 4,TH,EK,XC
    4 FORMAT(1H ,X,5HTHETA,3X,1HK,3X,4HLOC.,/,1H ,3F8.4,
     C//)
      DO 20 I=1,N
      Y=RANF(X)
      Z=-(ALOG(1.0-Y))
      T(I)=(TH*Z**(1./EK))+XC
   20 CONTINUE
   22 ML=1
      MHT=N-1
      ISW=1
      DO 50 I=ML,MHT
      IF(T(I)-T(I+1))50,50,25
   25 GO TO (30,40),ISW
   30 ISW=2
   40 TEMP=T(I)
      T(I)=T(I+1)
      T(I+1)=TEMP
   50 CONTINUE
      GO TO(55,22),ISW
   55 CONTINUE
      PRINT 56
   56 FORMAT(1H ,4X,26HCOMPONENT TIMES TO FAILURE,/)
      DO 60 I=1,N
      PRINT 57,T(I)
   57 FORMAT(1H ,F30.5)
   60 CONTINUE
      RETURN
      END
```

Vita

Robert Grantham Lannon, the son of Robert E. and Marian G. Lannon, was born on 28 October 1940, in Syracuse, New York. He was graduated from Jamesville-DeWitt High School, Lyndon, New York, in 1958 and entered the United States Air Force Academy that same year. He was graduated from the academy in June 1962 with a bachelors degree in Military Science and a commission as a 2nd Lieutenant in the USAF. After attending pilot training, he served in a variety of flying assignments before entering the Air Force Institute of Technology in June 1970.

Permanent Address: 106 Alwyn Rd.
DeWitt, New York 13214

This thesis was typed by Jane Manemann.